Homework Problems IV
PHYS 425: Electromagnetism I

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1. Griffiths Problem 4.12   Hint: when the constant polarization vector is "taken out" of the integral, what remains is the same integral as that needed in the "field of a sphere with uniform charge density".

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\boldsymbol{P}\cdot\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}dV' = \frac{\boldsymbol{P}\cdot}{4\pi\varepsilon_0}\int_S \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}dV'$$

Recall for a sphere of uniform charge density and total charge Q, the electric field is

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\int_S \frac{3Q}{4\pi R^3}\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}dV' = \begin{cases} \dfrac{Qr}{4\pi\varepsilon_0 R^3}\hat{r} & r < R \\ \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{r} & r > R \end{cases}$$

Consequently

$$\frac{1}{4\pi\varepsilon_0}\int_S \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}dV' = \begin{cases} \dfrac{r}{3\varepsilon_0}\hat{r} & r < R \\ \dfrac{R^3}{3\varepsilon_0 r^2}\hat{r} & r > R \end{cases}$$

So

$$V(r) = \begin{cases} \dfrac{r}{3\varepsilon_0}P\cdot\hat{r} & r < R \\ \dfrac{R^3}{3\varepsilon_0 r^2}P\cdot\hat{r} & r > R \end{cases}$$

$$V(r) = \begin{cases} \dfrac{rP}{3\varepsilon_0}\cos\theta & r < R \\ \dfrac{R^3 P}{3\varepsilon_0 r^2}\cos\theta & r > R \end{cases}$$

2. Griffiths Problem 4.18

   a) Inside both slabs $\boldsymbol{D}$ is pointing down and the magnitude is given by Gauss's Law as (ideal capacitor with no field leaking out the sides!)

$$-DA = \sigma_f A$$

$$\boldsymbol{D}_1 = \boldsymbol{D}_2 = -\sigma_f \hat{z}$$

b) The electric field is

$$\boldsymbol{E}_1 = \frac{\boldsymbol{D}_1}{\varepsilon_0 2} = -\frac{\sigma_f}{\varepsilon_0 2}\hat{z}$$
$$\boldsymbol{E}_2 = \frac{\boldsymbol{D}_2}{\varepsilon_0 1.5} = -\frac{\sigma_f 2}{\varepsilon_0 3}\hat{z}$$

c) The polarization is

$$\boldsymbol{P}_1 = \boldsymbol{D}_1 - \varepsilon_0 \boldsymbol{E}_1 = -\frac{\sigma_f}{2}\hat{z}$$
$$\boldsymbol{P}_2 = \boldsymbol{D}_2 - \varepsilon_0 \boldsymbol{E}_2 = -\frac{\sigma_f}{3}\hat{z}$$

d) The potential difference between the plates is

$$\Delta V = -E_1 a - E_2 a = +\frac{\sigma_f}{\varepsilon_0 2} + \frac{\sigma_f 2}{\varepsilon_0 3}$$

e) As ***D*** is uniform, there is no volume bound charge. There is surface bound charge at three locations: on the upper plate the bound surface charge is $-\sigma_f / 2$, on the lower plate the surface bound charge is $\sigma_f / 3$, and on the surface between the two slabs the surface bound charge is $\sigma_f / 2 - \sigma_f / 3 = \sigma_f / 6$.

f) Using all the charge, bound and free, to compute the electric field. Gauss's Law on the top and bottom plates gives

$$\boldsymbol{E}_1 = -\frac{\sigma_f}{\varepsilon_0}\left(1 - \frac{1}{2}\right)\hat{z} = -\frac{\sigma_f}{\varepsilon_0 2}\hat{z}$$
$$\boldsymbol{E}_2 = -\frac{\sigma_f}{\varepsilon_0}\left(1 - \frac{1}{3}\right)\hat{z} = -\frac{\sigma_f 2}{\varepsilon_0 3}\hat{z}$$

On the surface between the slabs, Gauss's Law also gives, consistent with above

$$\boldsymbol{E}_2 - \boldsymbol{E}_1 = -\frac{\sigma_f}{\varepsilon_0}\left(\frac{1}{6}\right)\hat{z} = -\frac{\sigma_f}{\varepsilon_0}\left(\frac{2}{3} - \frac{1}{2}\right)\hat{z}$$

3. Griffiths Problem 4.26

By Gauss's Law

$$\boldsymbol{D} = \frac{Q}{4\pi r^2}$$

The electric field is

$$\boldsymbol{E} = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0 r^2 (1+\chi_e)}\hat{r} & b > r > a \\ \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{r} & r > b \end{cases}$$

The total energy is

$$W = \frac{1}{2}\int \boldsymbol{D}\cdot\boldsymbol{E}dV$$
$$= \frac{Q^2}{8\pi\varepsilon_0}\left[\int_a^b \frac{r^2}{\varepsilon_{rel} r^4}dr + \int_b^\infty \frac{r^2}{r^4}dr\right]$$
$$= \frac{Q^2}{8\pi\varepsilon_0}\left[\frac{1}{\varepsilon_{rel}}\left(\frac{1}{a}-\frac{1}{b}\right)+\frac{1}{b}\right] = \frac{Q^2}{8\pi\varepsilon_0(1+\chi_e)}\left[\frac{1}{a}+\frac{\chi_e}{b}\right]$$

4. Griffiths Problem 5.3

g) The condition of zero deflection is the same as no net Lorentz force. Thus if the velocity is in the *z*-direction and the magnetic field in the *y*-direction,

$$E_x - v_z B_y = 0$$
$$v_z = \frac{E_x}{B_y}.$$

Such and arrangement of fields can act as a velocity filter for charged particles, called a Wien filter.

h) From Equation 5.10 in Griffiths,

$$v = \omega R = \frac{eB_y}{m}R = \frac{E_x}{B_y}$$
$$\frac{e}{m} = \frac{E_x}{B_y^2 R},$$

assuming the same velocity went into the magnetic field as before.

5. Griffiths Problem 5.6

a) If the record rotates with angular frequency ω, an individual point on the phonograph record traces a circle of circumference $2\pi r$. The time it takes to make one revolution is $2\pi/\omega$, and so the velocity is

$$v = \frac{2\pi r}{(2\pi / \omega)} = \omega r.$$

The surface current density is $K = \sigma v = \sigma \omega r$.

b) The volume charge density is

$$\rho = \frac{3Q}{4\pi R^3}$$

and the velocity, as in a) above, is $\omega\sqrt{x^2+y^2} = \omega r \sin\theta$ because $\sqrt{x^2+y^2}$ is the distance from the rotation axis. In vector form this result is $\boldsymbol{\omega}\times\boldsymbol{r}$. So the expression in polar coordinates is

$$\boldsymbol{J} = \rho \boldsymbol{v} = \frac{3Q}{4\pi R^3}\omega r \sin\theta\hat{\phi}$$

6. Griffiths Problem 5.25

Colin's solution is the correct one. It's just not obviously so! Note that he said that the solution is

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0 I}{4\pi}\int_{\theta_1}^{\theta_2}\frac{d\theta'}{\cos\theta'} = \frac{\mu_0 I}{4\pi}\hat{z}\ln\left(1/\cos\theta' + \tan\theta'\right)\Big|_{\theta_1}^{\theta_2}$$

$$\sin\theta' = \frac{z-z'}{\sqrt{x^2+y^2+(z-z')^2}}$$

$$\cos\theta' = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+(z-z')^2}}$$

Using $\cos\theta = \sqrt{(1+\sin\theta)(1-\sin\theta)}$, this solution becomes (See also Gradshteyn and Ryzhik Integral Tables Formula 2.562.9)

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0 I}{4\pi}\int_{\theta_1}^{\theta_2}\frac{d\theta'}{\cos\theta'} = \frac{\mu_0 I}{8\pi}\hat{z}\ln\left(\frac{1+\sin\theta'}{1-\sin\theta'}\right)\Bigg|_{\theta_1}^{\theta_2}$$

Once it is in this form, it is easy to see the correct solution emerges. When taking the derivatives in the curl expression for $\boldsymbol{B}$

$$\frac{\partial}{\partial x}\left[\frac{\mu_0 I}{8\pi}\ln\left(\frac{1+\sin\theta'}{1-\sin\theta'}\right)\right] = \frac{\mu_0 I}{8\pi}\left[\frac{\cos\theta'}{1+\sin\theta'}+\frac{\cos\theta'}{1-\sin\theta'}\right]\frac{\partial\theta'}{\partial x}$$

$$= \frac{\mu_0 I}{8\pi}\left[\frac{\cos\theta'(1-\sin\theta')+\cos\theta'(1+\sin\theta')}{1-\sin^2\theta'}\right]\frac{\partial\theta'}{\partial x}$$

$$= \frac{\mu_0 I}{4\pi}\left[\frac{1}{\cos\theta'}\right]\frac{\partial\theta'}{\partial x}$$

Implicit differentiation yields

$$\cos\theta' \frac{\partial\theta'}{\partial x} = -\frac{(z-z')x}{\left(x^2+y^2+(z-z')^2\right)^{3/2}}$$

$$\frac{\partial\theta'}{\partial x} = -\frac{\sin\theta'}{\cos\theta'}\frac{x}{x^2+y^2+(z-z')^2}$$

So indeed

$$\boldsymbol{B} = \nabla\times\boldsymbol{A} = -\frac{\mu_0 I}{4\pi}\left[\frac{\sin\theta'}{\cos^2\theta'}\right]\frac{y\hat{x}-x\hat{y}}{x^2+y^2+(z-z')^2}\Bigg|_{\theta_1}^{\theta_2}$$

$$= -\frac{\mu_0 I}{4\pi s^2}\sin\theta'\Big|_{\theta_1}^{\theta_2}(y\hat{x}-x\hat{y}),$$

Exactly, the 5.37 result! (Note: $(x\hat{y}-y\hat{x})/s=\hat{\phi}$)

It is possible to get the result using Colin's original expression.

Differentiating Colin's expression

$$\frac{\partial}{\partial x}\ln(1/\cos\theta'+\tan\theta') = \frac{\partial}{\partial x}\ln\left((1+\sin\theta')/\cos\theta'\right)$$

$$= \frac{\cos\theta'}{1+\sin\theta'}\left[\frac{\cos\theta'}{\cos\theta'}+\frac{(1+\sin\theta')\sin\theta'}{\cos^2\theta'}\right]\frac{\partial\theta'}{\partial x}$$

$$= \frac{\cos\theta'}{1+\sin\theta'}\left[\frac{\cos^2\theta'+\sin\theta'+\sin^2\theta'}{\cos^2\theta'}\right]\frac{\partial\theta'}{\partial x}$$

$$= \frac{1}{\cos\theta'}\frac{\partial\theta'}{\partial x}$$

The rest of the derivation follows the above.

It turns out, as shown in Maggie Bragg's solution, that an easy solution also comes from the expression

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0 I}{4\pi}\hat{z}\int_{z_1}^{z_2}\frac{dz'}{\sqrt{s^2+(z-z')^2}} = \frac{\mu_0 I}{4\pi}\hat{z}\left[-\ln\left(\frac{z-z'+\sqrt{s^2+(z-z')^2}}{s}\right)\right]_{z_1}^{z_2}$$

$$= \frac{\mu_0 I}{4\pi}\hat{z}\ln\left(\frac{z-z_1+\sqrt{s^2+(z-z')^2}}{z-z_2+\sqrt{s^2+(z-z')^2}}\right)$$

Indeed

$$\nabla \times \boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0 I}{4\pi}\left(\hat{x}\frac{\partial}{\partial y} - \hat{y}\frac{\partial}{\partial x}\right)\ln\left(\frac{z - z_1 + \sqrt{s^2 + (z - z')^2}}{z - z_2 + \sqrt{s^2 + (z - z')^2}}\right)$$

$$\frac{\partial}{\partial y}\ln\left(\frac{z - z_1 + \sqrt{s^2 + (z - z_1)^2}}{z - z_2 + \sqrt{s^2 + (z - z_2)^2}}\right) = \frac{1}{z - z_1 + \sqrt{s^2 + (z - z_1)^2}}\frac{y}{\sqrt{s^2 + (z - z_1)^2}} - \frac{y}{s^2} - (z_1 \to z_2)$$

$$= \frac{y\left(s^2 - s^2 - (z - z_1)^2 - (z - z_1)\sqrt{s^2 + (z - z_1)^2}\right)}{\left(z - z_1 + \sqrt{s^2 + (z - z_1)^2}\right)\sqrt{s^2 + (z - z_1)^2}\, s^2} - (z_1 \to z_2)$$

$$= -\frac{y(z - z_1)}{\sqrt{s^2 + (z - z_1)^2}\, s^2} + \frac{y(z - z_2)}{\sqrt{s^2 + (z - z_2)^2}\, s^2} = -\frac{y}{s^2}(\sin\theta_1 - \sin\theta_2)$$

Final comment. To show the correct magnetic field results from the integral expression, one can differentiate inside the integral and make a trigonometric substitution

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0 I}{4\pi}\hat{z}\int_{z_1}^{z_2}\frac{dz'}{\sqrt{s^2 + (z - z')^2}}$$

$$\nabla \times \boldsymbol{A} = \frac{\mu_0 I}{4\pi}\left(\hat{x}\frac{\partial}{\partial y} - \hat{y}\frac{\partial}{\partial x}\right)\int_{z_1}^{z_2}\frac{dz'}{\sqrt{s^2 + (z - z')^2}}$$

$$= -\frac{\mu_0 I}{4\pi}(\hat{x}y - \hat{y}x)\int_{z_1}^{z_2}\frac{dz'}{\left(s^2 + (z - z')^2\right)^{3/2}}$$

$$z' - z = s\tan\theta$$

$$= \frac{\mu_0 I}{4\pi}(\hat{y}x - \hat{x}y)\int_{\theta_1}^{\theta_2}\frac{s}{\cos^2\theta}\frac{\cos^3\theta\, d\theta}{s^3}$$

$$= \frac{\mu_0 I}{4\pi s^2}(\hat{y}x - \hat{x}y)(\sin\theta_2 - \sin\theta_1)$$